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An extension of Glimm's method to inhomogeneous strictly hyperbolic systems of conservation laws by "weaker than weak" solutions of the Riemann problem

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Abstract

We construct a generalized solution of the Riemann problem for strictly hyperbolic systems of conservation laws with source terms, and we use this to show that Glimm's method can be used directly to establish the existence of solutions of the Cauchy problem. The source terms are taken to be of the form $a'G$, and this enables us to extend the method introduced by Lax to construct general solutions of the Riemann problem. Our generalized solution of the Riemann problem is "weaker than weak" in the sense that it is weaker than a distributional solution. Thus, we prove that a weak solution of the Cauchy problem is the limit of a sequence of Glimm scheme approximate solutions that are based on "weaker than weak" solutions of the Riemann problem. By establishing the convergence of Glimm's method, it follows that all of the results on time asymptotics and uniqueness for Glimm's method (in the presence of a linearly degenerate field) now apply, unchanged, to inhomogeneous systems.

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1. Introduction

We consider the $n \times n$ system of conservation laws

$$u_t + f(a,u)_x = a' g(a,u), \tag{1}$$

where $u = (u_1(x,t), \dots, u_n(x,t))$, $f = (f_1(a,u), \dots, f_n(a,u))$ and $g = (g_1(a,u), \dots, g_n(a,u))$ are smooth functions of (a,u). The variable $a = a(x)$ is assumed to be a *Lipschitz continuous* function of x that is of finite total variation, and $a' \equiv \frac{da}{dx}$. Note that when $a(x) = x$, system (1) is a general inhomogeneous system of the form

$$u_t + f(x,u)_x = g(x,u),$$

but then $a(x) = x$ is of infinite total variation. An important special case of such a system is the system of compressible Euler equations in a variable area duct,

$$\begin{aligned}
\rho_t + (\rho u)_x &= -\frac{a'(x)}{a(x)}\rho u,\\
(\rho u)_t + (\rho u^2 + p)_x &= -\frac{a'(x)}{a(x)}\rho u^2,\\
(\rho E)_t + (\rho E u + p u)_x &= -\frac{a'(x)}{a(x)}(\rho E u + p u),
\end{aligned}$$

where ρ, u represent the density and velocity of a fluid, p and E represent the pressure and total energy, and the variable a denotes the area of the variable duct.

Following the lead in LeFloch [14], Isaacson and Temple [9], we augment system (1) by adding the equation $a_t = 0$. We then obtain the equivalent $(n+1) \times (n+1)$ system of conservation laws,

$$U_t + F(U)_x = a' G(U), \tag{2}$$

where $U \equiv (a, u_1, \dots, u_n)$, $F(U) \equiv (0, f_1(U), \dots, f_n(U))$, and $G(U) \equiv (0, g_1(U), \dots, g_n(U))$.

In this paper we consider the Cauchy problem for system (2),

$$\begin{cases} U_t + F(U)_x = a' G(U),\\ U(x,0) = U_0(x), \end{cases} \tag{3}$$

where $U_0(x)$ denotes the initial data. The Riemann problem for system (2) is the Cauchy problem with the piecewise constant initial data

$$U(x,0) = \begin{cases} U_{\mathrm{L}}, & x < 0,\\ U_{\mathrm{R}}, & x > 0, \end{cases}$$

where U_L, U_R are two constant states. The purpose of this paper is to show that the solution of the Riemann problem can be used in a Glimm scheme as building blocks for the construction of general weak solutions of the Cauchy problem (3). This is slightly surprising because the approximation of $a(x)$ by piecewise constant states does not provide a uniform approximation for $a'(x)$ which appears in Eq. (2). Nevertheless, in this paper we prove that this procedure is valid.

For smooth solutions, system (2) is equivalent to

$$U_t + DF(U) \cdot U_x = a' G(U),$$

where the Jacobian matrix $DF(U)$ is given by

$$DF(U) \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \frac{\partial f_1}{\partial a} & \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial a} & \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_n} \end{pmatrix}.$$

Let $\lambda_0(U) \equiv 0$ denote the zero eigenvalue of Df, let $\{\lambda_1, \ldots, \lambda_n\}$ denote the remaining eigenvalues of $DF(U)$, and let $\{r_i(U); 0 \leqslant i \leqslant 1\}$, $\{\mathrm{l}_i(U); 0 \leqslant i \leqslant 1\}$ denote the right and left eigenvectors of $DF(U)$ with respect to $\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$. In this paper we assume that system (2) is strictly hyperbolic, that is, $\lambda_1 < \cdots < \lambda_{k-1} < \lambda_0 = 0 < \lambda_k < \cdots < \lambda_n$.

The Cauchy problem for the quasilinear hyperbolic system

$$U_t + F(x, U)_x = G(x, U)$$

was first studied by Liu in 1979 [17]. In this foundational paper, the approximate solution of the Cauchy problem was constructed by solving the steady-state solution in each time step. Liu showed that the global solution exists and tends pointwise to a steady solution when the L^1 norm of $G(x, U)$ and $\frac{\partial G}{\partial U}$ and the total variation of the initial data is small. In this paper, using the structure of the source terms, we are able to use the Glimm scheme to prove the existence of solutions of the Cauchy problem (3). Picking up on the idea in [9], we reformulate the source term so that a rescaling argument can be used, and using this, we show that Lax's method can be applied to obtain the general solution of the Riemann problem that accounts for the discontinuities in the source term, as well as the conserved quantities. Finally, we demonstrate that, when $a(x)$ is Lipschitz-continuous, the residual of the approximate solution converges weakly (that is, *by oscillation*) to zero, and this implies that the limit function U, the function extracted from the compactness of the approximation scheme, is a veritable weak solution of system (3). In contrast, in Glimm's original paper, the residual is shown to converge to zero by strong L^1 convergence. Since our scheme employs only a pointwise approximation of $a(x)$, it follows that we cannot expect a strong convergence of $a' G(U)$. Since the residual involves only the integral of

$a'G(U)$, the weak convergence of $a'G(U)$ is sufficient to prove that the limit function is a weak solution. In Section 4 we prove the weak convergence of the residual, which is perhaps the most interesting and surprising point in the paper. The convergence of Glimm's method can then be achieved directly. By this method we extend Glimm's method to inhomogeneous systems.

We find it interesting that although the Glimm method is based on solutions of the Riemann problem that are not distributional solutions of the equations, the limit solutions are true weak solutions when a is Lipschitz continuous. Since the Riemann problem is constructed from $n+1$ eigenvector fields as in the Lax construction, it follows that all of the results on the time asymptotics and uniqueness of solutions for Glimm's method with a linearly degenerate field apply unchanged to inhomogeneous systems of form (3). A general theory of making sense of such weaker than distributional solutions of non-conservative systems was carried out by LeFloch [1,13–15]. In these papers, the source terms of these systems were described based on a family of Lipschitz paths, and the product of a' and $G(U)$ was defined as a Borel measure (a non-conservative product of $\frac{da}{dx}$ and $G(U)$). The existence result can be obtained followed by this framework. Our analysis here does not require any more information about the Riemann problem other than its construction.

In the case of non-strictly hyperbolic systems, the existence result by Glimm's method for 2×2 homogeneous systems was first established by Temple [21]. Here we can see that the technique demonstrated in this paper can also be applied to the 2×2 resonant systems with source terms as described in (1), cf. [7]. For more details on resonant systems, we refer the reader to [4,6,8,9].

We assume that each characteristic field is either genuinely non-linear or linearly degenerate. Our goal is to extend Glimm's method to prove the global existence of the weak solutions of the Cauchy problem for an inhomogeneous system. For the steps in the proof corresponding to the steps employed by Glimm in the solutions of the Cauchy problem for homogeneous system, cf. [2],

$$\begin{cases} U_t + F(U)_x = 0, \\ U(x,0) = U_0(x). \end{cases} \tag{4}$$

Glimm showed that if the total variation of $U_0(x)$ is sufficiently small, then the weak solution will exist for all time $t > 0$. Glimm's proof employed the following steps:

(I) The construction of the approximate solution for the Cauchy problem.
(II) The wave interaction estimate.
(III) The decreasing of the total variation for the approximate solution.
(IV) The compactness of the approximate solution.

In this paper, we extend steps (I)–(IV) of Glimm's method so that the argument applies to general inhomogeneous system (2). We start at the homogeneous system corresponding to system (2). Since one of the eigenvalues of $DF(U)$ is identically zero, the characteristic field with respect to this eigenvalue is linear degenerate, and so we have a contact discontinuity in the solution of the Riemann problem. For the inhomogeneous system (2), we introduce a new kind of waves called standing wave discontinuities which correspond to the characteristic field with zero eigenvalue. For the general Riemann problem of the inhomogeneous system we have four different kinds of waves: shocks wave, rarefaction waves, contact discontinuities and standing wave discontinuity, which appear as elementary waves in the solution of the Riemann problem. By Lax's method we prove the existence of the solution of the Riemann problem, and this solution consists of constant states separated by elementary waves.

For the Cauchy problem we use Glimm's method to construct the approximate solution and obtain the L^1_{loc}-compactness of the approximate solution. But because of the presence of the source term $a'G(U)$, the function to which the approximate solution converges, will not satisfy the definition of the weak solution for the Cauchy problem. The idea to overcome the difficulty is the following: first we denote the approximate solutions of Cauchy problem (3) by the Glimm scheme as $\{U_{\theta,\triangle x}\}$. The new approximate solutions $\{U^{\varepsilon}_{\theta,\triangle x}\}$ of Cauchy problem (3) can be constructed by smoothing out standing wave discontinuities in $\{U_{\theta,\triangle x}\}$ by smooth standing waves. Here we require the function a_{ε} in the smooth standing waves $(a_{\varepsilon}, u_{\varepsilon})$ in $\{U^{\varepsilon}_{\theta,\triangle x}\}$ as a monotone function to be the new condition we impose. We obtain the following properties of $\{U^{\varepsilon}_{\theta,\triangle x}\}$ for all $\varepsilon > 0$.

1. $\{U^{\varepsilon}_{\theta,\triangle x}\}$ is uniformly bounded.
2. $\{U^{\varepsilon}_{\theta,\triangle x}\}$ has bounded total variation.
3. $\{U^{\varepsilon}_{\theta,\triangle x}\}$ is *Lipschitz-continuous* with respect to time t.

And then we can use Helly's theorem to prove the convergence of $\{U^{\varepsilon}_{\theta,\triangle x}\}$. This means that there exists a subsequence of $U^{\varepsilon}_{\theta,\triangle x}$ that converges in L^1_{loc} to some function $U(x,t)$ as ε and $\triangle x$ approach 0.

We define the residual $R_{\phi}(U)$ as

$$
\begin{aligned}
R_{\phi}(U) = \int\int_{t>0} & U\phi_t + F(U)\phi_x + a'G(U)\phi \, dx\, dt \\
& + \int_{-\infty}^{\infty} U_0(x)\phi(x,0)\, dx, \quad \phi \in C_0^1(R^2).
\end{aligned}
$$

A bounded measurable function $U(x,t)$ is a weak solution of the Cauchy problem (3) if and only if $U(x,t)$ satisfies $R_{\phi}(U(x,t))$=0. Then by the construction of the $\{U^{\varepsilon}_{\theta,\triangle x}\}$, the residual $R_{\phi}(U^{\varepsilon}_{\theta,\triangle x})$ is an integrable function of ε and $\triangle x$. In Section 4 we prove

$$
\lim_{\varepsilon\to 0,\triangle x\to 0} R_{\phi}(U^{\varepsilon}_{\theta,\triangle x}) = 0
$$

for almost all choices of θ. Then by the Lebesgue bounded convergence theorem, we have

$$0 = \lim_{\varepsilon\to 0, \triangle x\to 0.} R_\phi(U^\varepsilon_{\theta,\triangle x}) = R_\phi\left(\lim_{\varepsilon\to 0, \triangle x\to 0.} U^\varepsilon_{\theta,\triangle x}\right) = R_\phi(U(x,t))$$

for almost all choices of θ. So we have the following main theorem of the paper.

Main Theorem. *Consider the Cauchy problem of the strictly hyperbolic system* (3). *If the total variation of $U_0(x)$ is sufficiently small, then there exists a null set $N \subset \Phi$ and a sequence $\Delta x_i \to 0$ such that if $\theta \in \Phi \setminus N$,*

$$U(x,t) \equiv \lim_{\varepsilon\to 0, \Delta x_i\to 0} U^\varepsilon_{\theta,\Delta x_i}$$

is a weak solution of the Cauchy problem (3).

2. The Riemann problem

In this section, we study the following Riemann problem of the $(n+1)\times(n+1)$ strictly hyperbolic system of conservation laws:

$$U_t + F(U)_x = a'G(U), \qquad U(x,0) = \begin{cases} U_{\mathrm{L}}, & x < 0, \\ U_{\mathrm{R}}, & x > 0 \end{cases} \tag{5}$$

or

$$U_t + DF(U)U_x = a'G(U), \quad U(x,0) = \begin{cases} U_{\mathrm{L}}, & x < 0, \\ U_{\mathrm{R}}, & x > 0, \end{cases} \tag{6}$$

where U_{L} and U_{R} are two nearby constant states. The Jacobian matrix $DF(U)$ is defined in the Introduction. We assume that the eigenvalues $\{\lambda_0, \ldots, \lambda_n\}$ of $DF(U)$ satisfy

$$\lambda_1 < \cdots < \lambda_{k-1} < \lambda_0 = 0 < \lambda_k < \cdots < \lambda_n$$

and each characteristic field is either genuinely non-linear or linearly degenerate. We prove the existence of weak solutions of the Riemann problem (5) for a system of conservation law by Lax's method. To begin, we review Lax's construction in the homogeneous case when $g \equiv 0$.

The Riemann problem of homogeneous systems is the initial value problem when the initial data consist of a jump discontinuity

$$U_t + F(U)_x = 0, \quad U(x,0) = \begin{cases} U_{\mathrm{L}}, & x < 0, \\ U_{\mathrm{R}}, & x > 0. \end{cases} \tag{7}$$

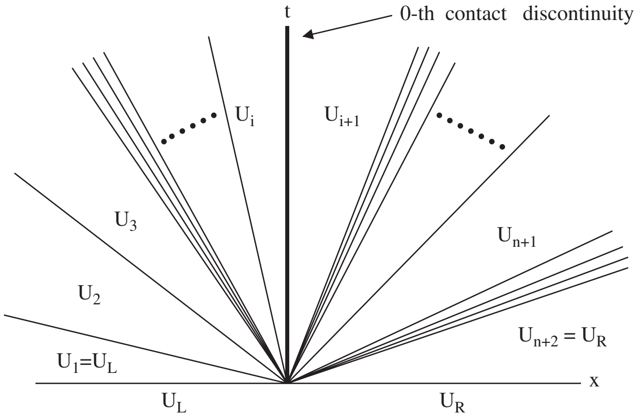

Fig. 1. The solution for the homogeneous Riemann problem.

The Riemann problem (7) was first studied by Lax in the 1950s [11]. The solution is constructed by connecting U_L and U_R by wave curves in phase space; there is a wave curve corresponding to each type of elementary wave: rarefaction wave, shock wave and contact discontinuity. Since there is more than one way to connect U_L and U_R by these wave curves, the Lax entropy condition is required to be added to rule out unphysical possible weak solutions. Lax proved that the solution of the Riemann problem (7) exists uniquely within the class of entropy-satisfying waves, as long as U_L is sufficiently close to U_R. The solution consists of constant states separated by those three kinds of waves. We state the following theorem which is due to Lax: (see [20]).

Theorem 2.1. *Consider the Riemann problem* (7). *Assume that the system is strictly hyperbolic and that each characteristic field is either genuinely non-linear or linearly degenerate. Then for each constant state $U_L \in D$, the Riemann problem has a unique solution for each constant state U_R sufficiently close to U_L. The solution consists of* $(n+2)$ *constant states separated by shocks, centered simple waves and contact discontinuities, and the zero family is a contact discontinuity of zero speed.*

Fig. 1 describes the solution of the Riemann problem of the homogeneous system (7).

For the inhomogeneous systems we use the elementary waves in the homogeneous cases along with the idea of standing wave discontinuities to describe the solution of a Riemann problem for the inhomogeneous system (2), and prove the existence and uniqueness of the entropy solutions.

First, we study the smooth time-independent (standing wave) solutions of system (2). Such solutions exist since one of the eigenvalues of $DF(U)$ is zero. The time-independent solutions $U \equiv (a(x), u_1(x), \dots, u_n(x))$ in (5) satisfy the following system of ordinary differential equations:

$$f_i(a,u)_x = a' g_i(a,u), \;\; i = 1,\dots,n.$$

By chain rule we have the following system:

$$\frac{\partial f_i}{\partial a}\frac{da}{dx} + \sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial u_j}\right)\frac{du_j}{dx} = g_i\frac{da}{dx}, \qquad i = 1,\dots,n. \tag{8}$$

This gives us an $n\times n$ system of ordinary differential equations with $n+1$ variables. Since $\lambda_i \neq 0$ for $i \neq 0$, the condition

$$\det\left(\frac{\partial f_i}{\partial u_j}\right) \neq 0 \tag{9}$$

holds. Here we assume that the vector

$$\left(\frac{\partial f}{\partial u}\right)^{-1}\left(g - \frac{\partial f}{\partial a}\right)(U) \neq 0 \quad \text{for } U \in \Omega. \tag{10}$$

For example, in the system of compressible Euler equations (we omit the equation of total energy), the vector

$$\left(\frac{\partial f}{\partial u}\right)^{-1}\left(g - \frac{\partial f}{\partial a}\right) = \left(\frac{\partial f}{\partial u}\right)^{-1} g = \begin{pmatrix} -\left(\frac{m^2}{\rho^2} - p'\right)^{-1}\frac{\rho u^2}{a} \\ -\frac{\rho u}{a} \end{pmatrix} \neq 0$$

for all $a, \rho > 0$ and $u \neq c \equiv \sqrt{p'}$.

Next, by the Lipschitz continuity and the monotonicity of a ($\frac{da}{dx} \neq 0$), we can divide Eq. (10) by $\frac{da}{dx}$ and obtain the following system:

$$\sum_{j=1}^{n}\left(\frac{\partial f_i}{\partial u_j}\right)\frac{\partial u_j}{\partial a} = g_i - \frac{\partial f_i}{\partial a}, \quad i = 1,\dots,n. \tag{11}$$

Under assumptions (9), (10) we can solve system (11) for $\frac{\partial u_j}{\partial a}$, $j = 1,\dots,n$. This means that we can express each u_j as a smooth function of a for $j = 1,\dots,n$. Now, we define the n-vector $r_0(U)$ as

$$r_0(U) \equiv \left(\frac{\partial f}{\partial u}\right)^{-1}\left(g - \frac{\partial f}{\partial a}\right). \tag{12}$$

This defines a vector field $R_0(U)$ on U-space by

$$R_0(U) \equiv \begin{pmatrix} 1 \\ r_0(U) \end{pmatrix}. \tag{13}$$

Note that $R_0(U)$ is linearly independent of the eigenvectors $\{R_1, \ldots, R_n\}$ of $DF(U)$, where

$$R_i(U) \equiv \begin{pmatrix} 0 \\ r_i(U) \end{pmatrix} \tag{14}$$

and $r_i(U)$ is an eigenvector of the $n \times n$ matrix $(\frac{\partial f_i}{\partial u_j})$.

Let $U_s(\varepsilon)$ denote the integral curve of $R_0(U)$ through U_L, parameterized by $\varepsilon = a$. Then we can use $U_s(\varepsilon)$ to construct standing wave solutions of system (2) as follows: let $a_\varepsilon(x)$ be any smooth *monotone* function taking a_L to a_R as x ranges from $-\varepsilon$ to ε. For example,

$$a_\varepsilon(x) = a_L + \phi_\varepsilon(x) \cdot (a_L - a_R) \tag{15}$$

takes a_L to a_R as x ranges from $-\varepsilon$ to ε, where $\phi_\varepsilon(x)$ is a smooth monotone function going from 0 to 1 as x ranges from $-\varepsilon$ to ε. From (12) we know that $U_s(\varepsilon) = (a_\varepsilon(x), u_s(a_\varepsilon(x)))$ is a smooth standing wave if

$$\frac{du_s}{da_\varepsilon} = \left(\frac{\partial f}{\partial u_s}\right)^{-1}\left(g - \frac{\partial f}{\partial a_\varepsilon}\right) = r_0(U_s).$$

It is equivalent to $\dot{U}_s(\varepsilon) = R_0(U_s(\varepsilon))$ if we choose a_ε as the parameter of U_s. If there exist some points where the smooth function a_ε is not monotone, then at those points the standing wave $U_s(\varepsilon)$ cannot match the integral curve of R_0. So it is necessary for us to give the condition for a_ε

$$\textit{The function } a_\varepsilon(x) \textit{ in } U_s(\varepsilon) \textit{ is always monotone.} \tag{16}$$

Under condition (16), the function $U_s(a_\varepsilon)$ is a standing wave of system (2) if U_s is an integral curve of R_0, and can also be parameterized as

$$U_s(\varepsilon) = U_L + \varepsilon R_0(U_s(\varepsilon)) + \frac{\varepsilon^2}{2} R_0 \cdot \nabla R_0(U_s(\varepsilon)) + O(\varepsilon^3). \tag{17}$$

Now we consider the standing waves on the (x,t)-plane. If the function

$$U_s^\varepsilon(x) \equiv (a_\varepsilon(x), u_{1s}(a_\varepsilon(x)), \ldots, u_{ns}(a_\varepsilon(x)))$$

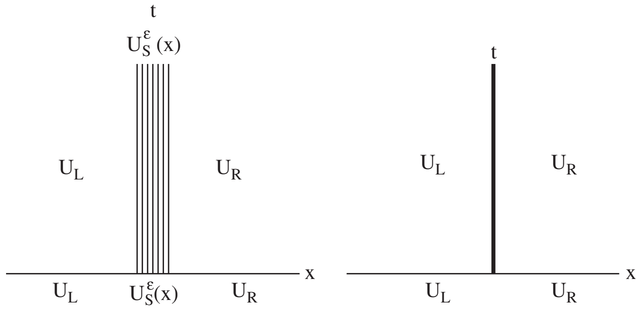

Fig. 2. The smooth standing wave and standing wave discontinuity.

connecting U_{L} and U_{R} is a standing wave of system (2), where $a_\varepsilon(x)$ is a given smooth curve that connects a_{L} and a_{R} within $-\varepsilon < x < \varepsilon$, then the associated initial data of $U_{\mathrm{s}}^{\varepsilon}(x)$ will be given by

$$U_0^{\varepsilon}(x) = \begin{cases} U_{\mathrm{L}}, & x < -\varepsilon, \\ (a_\varepsilon(x), u_{1\mathrm{s}}(a_\varepsilon(x)), \ldots, u_{n\mathrm{s}}(a_\varepsilon(x))), & -\varepsilon \leqslant x \leqslant \varepsilon, \\ U_{\mathrm{R}}, & x > \varepsilon. \end{cases}$$

To let $U_0^{\varepsilon}(x)$ match the initial data $U_0(x)$, we require the parameter ε to approach zero. In this case, $a_\varepsilon(x)$ becomes discontinuous, which means that $U_{\mathrm{s}}^{\varepsilon}(x)$ becomes a discontinuous function as ε approaches zero.

Definition 2.1. A discontinuous function $U_{\mathrm{s}}(x)$ is called a standing wave discontinuity of the Riemann problem (5) if $U_{\mathrm{s}}(x)$ is the limit of a sequence of smooth standing waves $U_{\mathrm{s}}^{\varepsilon}(x) \equiv (a_\varepsilon(x), u_{1\mathrm{s}}(a_\varepsilon(x)), \ldots, u_{n\mathrm{s}}(a_\varepsilon(x)))$ of system (11).

By previous analysis we have the following theorem.

Theorem 2.2. *For the Riemann problem* (5), *the* 0*th characteristic field with respect to* $\lambda_0 = 0$ *is linearly degenerate in* Ω, *and if* $U_{\mathrm{L}} \in \Omega$, *then there exists a smooth parameter of states that can be connected to* U_{L} *on the right by the standing wave discontinuity*; *see Fig*. 2.

In the following, we see how the analysis of the Riemann problem in the homogeneous cases can be modified to apply to the Riemann problem in the inhomogeneous system (5) (Fig. 3). First since $a_t = 0$ and the initial data $U_0(x)$ for the Riemann problem (5) have a jump discontinuity in a at $x = 0$, it follows that $a(x)$ is constant for $x \neq 0$. Then the solution of the Riemann problem (5) at $x \neq 0$ actually solves the homogeneous system at constant a. Thus, for each characteristic field, $i = 1, \ldots, n$, we can use the i-wave from the homogeneous system as building blocks for the solution of the Riemann problem in the inhomogeneous system. These introduce i-shocks,

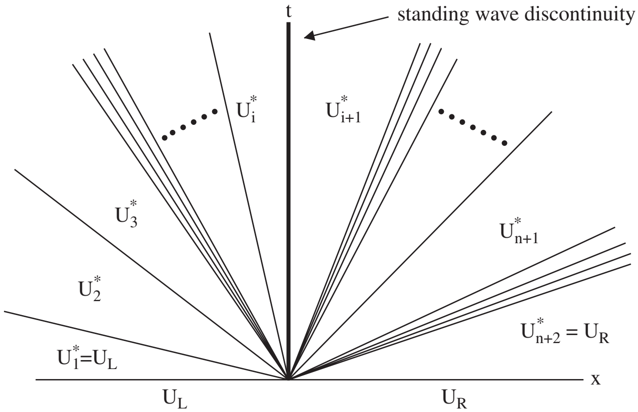

Fig. 3. The solution of the Riemann problem in the inhomogeneous case.

i-centered simple waves and i-contact discontinuities into the solution. This means that we will have four kinds of elementary waves in the solution of the Riemann problem in the inhomogeneous system: shocks, centered simple waves, contact discontinuities and standing wave discontinuities. So given $U_L \in \Omega$ we only need to show that a state U_R can be connected to U_L on the right by a sequence of such waves. We will use Lax's idea to prove that there exists a unique way to select those waves curves that connect U_L to U_R.

Now we prove the main theorem of this section. Before we prove the following theorem, we review the following vectors defined in the previous section:

$$R_0(U) \equiv \begin{pmatrix} 1 \\ r_0(U) \end{pmatrix}, \quad R_i(U) \equiv \begin{pmatrix} 0 \\ r_i(U) \end{pmatrix}, \quad i = 1, \dots, n.$$

Here $r_i(U)$ is an eigenvector of an $n \times n$ matrix $(\frac{\partial f_i}{\partial u_j})$, and $r_0(U)$ is defined in (12). Note that $R_0(U)$ is linearly independent of $\{R_1, \dots, R_n\}$.

Theorem 2.3. *Consider the Riemann problem* (5); *supposing that* U_L, $U_R \in \Omega$ *and* $|U_L - U_R|$ *is sufficiently small, each characteristic field is either genuinely non-linear or linearly degenerate. Then, we can find a neighborhood* $N \subset \Omega$ *such that if* U_L, $U_R \in N$, *then the Riemann problem* (5) *has a unique solution. The solution consists of at most* $(n+2)$ *constant states separated by shocks, centered simple waves, contact discontinuities and standing wave discontinuities.*

Proof. By the theorems in previous sections, given a constant state $U \in N$ we can always find a set of mappings $\{T^i_{\varepsilon_i};\ T^i_{\varepsilon_i} : N \to R^n,\ i = 0, \dots, n,\ |\varepsilon_i| < \beta$ for some $\beta.\}$; each $T^i_{\varepsilon_i}$ is at least a C^2 mapping, such that $T^i_{\varepsilon_i}(U)$ can be connected to U on the right by either shocks, centered simple waves, contact discontinuities or standing wave discontinuity. We define $W = \{(\varepsilon_0, \ldots, \varepsilon_n) \in R^n : |\varepsilon_i| < \beta.\}$. And consider the following mapping:

$$T_\varepsilon(U) \equiv T^n_{\varepsilon_n} \cdot T^{n-1}_{\varepsilon_{n-1}} \cdots T^1_{\varepsilon_1} \cdot T^0_{\varepsilon_0}(U),$$
$$\varepsilon \equiv (\varepsilon_0, \ldots, \varepsilon_n) \in W, \quad U \in N.$$

So our main goal is to prove that there exists a unique ε such that $T_\varepsilon(U_L) = U_R$.

By the previous analysis we know that $T^i_{\varepsilon_i}(U) = U + \varepsilon_i \cdot R_i(U) + O(\varepsilon_i^2)$ for the ith characteristic field, $i \neq 0$. And $T^j_{\varepsilon_j}(U) = U + \varepsilon_j \cdot R_0(U) + O(\varepsilon_j^2)$ for the 0th characteristic field. Without loss of generality we assume that ε_0 is the parameter corresponding to $R_0(U)$. Then by direct calculation we obtain

$$T_\varepsilon(U) = U + \sum_{k=1}^{n} \varepsilon_k R_k(U) + \varepsilon_0 R_0(U) + O(\varepsilon^2).$$

Define the mapping $F(\varepsilon)$ as

$$F(\varepsilon) = T_\varepsilon(U_L) - U_L.$$

Then $F(0) = 0$ and $F(\varepsilon) = \sum_{k=1}^{n} \varepsilon_k R_k(U_L) + \varepsilon_0 R_0(U_L) + O(\varepsilon^2)$, $\varepsilon = (\varepsilon_0, \ldots, \varepsilon_n) \in W$, and therefore the $(n+1) \times (n+1)$ Jacobian matrix

$$DF(0) = (R_0(U_L), R_1(U_L), \ldots, R_n(U_L))$$

is non-singular due to the linear independence of $\{R_0(U), R_1(U), \ldots, R_n(U)\}$. So the mapping $F(\varepsilon)$ is a diffeomorphism from a neighborhood of $\varepsilon = 0$ to a neighborhood of $U = U_L$. By the inverse function theorem, if we choose $|U_L - U_R|$ sufficiently small, then we can find a unique $\bar{\varepsilon}$ in the neighborhood of $\varepsilon = 0$ such that

$$F(\bar{\varepsilon}) = U_R - U_L.$$

This means that $T_{\bar{\varepsilon}}(U_L) = U_R$. Thus, we complete the proof. $\square$

3. Glimm's method for Cauchy problem

In this section, we first review Glimm's existence theorem for the Cauchy problem for systems of conservation laws. Glimm's original theorem applies to the case $g \equiv 0$, cf. [2]. To start, we consider the Cauchy problem for the $(n+1) \times (n+1)$ homogeneous system,

$$\begin{cases} U_t + F(U)_x = 0, \\ U(x, 0) = U_0(x), \end{cases} \tag{18}$$

where U and $F(U)$ are defined in the introduction. We assume that $U_0(x)$ is a function with a bounded total variation. In the fundamental paper of Glimm, cf. [2], the approximate solution of the Cauchy problem was constructed by approximating $U_0(x)$ by a step function and solving for a series of Riemann problems on the first time step. To continue the scheme, the initial data at a subsequent time step will be given by a random choice of data in the approximate solution at the previous time step. In this way, the approximate solution of the Cauchy problem is constructed inductively. To apply Glimm's analysis, it is well known that the main point is to demonstrate the bounded total variation and Lipschitz-continuity in time of the approximate solution. Then we can use Helly's theorem to obtain the L^1_{loc}-compactness for the subsequence of the approximate solution. So the most important step is to construct a non-increasing functional which is equivalent to the total variation of the approximate solution. To prove that the functional is non-increasing with respect to time we need to know the relationship between the Riemann problems at adjoint time steps, which involves the interaction of the waves.

The reason we discuss the Cauchy problem for the homogeneous case is to show that we can extend these results to the inhomogeneous case. Recall the previous section; the solution of the Riemann problem (5) consists of constant states separated by shock waves, simple waves, contact discontinuities and standing wave discontinuity. This means that the difference of the structure of waves between the homogeneous and inhomogeneous cases is in the presence of the 0th characteristic field, the field of standing wave discontinuities. The standing wave discontinuities are like contact discontinuities except that they are not weak solutions of the equations. Indeed, in both cases, the wave curves in the state space can be smoothly parameterized, that is, $U^0_\varepsilon(U) = U + \varepsilon \cdot \bar{R}_0(U) + O(\varepsilon^2)$ in the case of contact discontinuities (the vector $\bar{R}_0(U)$ is an eigenvector of $DF(U)$ associated with eigenvalue λ_0), and $U^0_{\bar{\varepsilon}}(U) = U + \bar{\varepsilon} \cdot R_0(U) + O(\bar{\varepsilon}^2)$ in the case of standing wave discontinuities. The functional we define depends on the parameter ε, so in the inhomogeneous case we just need to replace ε in the homogeneous case by $\bar{\varepsilon}$, and we obtain the same results as the homogeneous case.

We begin by defining the Glimm scheme precisely. First we divide the (x,t)-plane into

$$x_k = k\triangle x,\ \ t_i = i\triangle t,\ \ k = 0, \pm 1, \pm 2, \ldots,\ \ i = 0, 1, 2, \ldots\ .$$

We take the C.F.L. condition

$$\frac{\triangle x}{\triangle t} > 2 \cdot \max\{\lambda_j;\ j \in \{0, 1, \cdot, n\},$$

to avoid the interaction of waves on the same time level. (The factor 2 allows us to spread out the standing wave discontinuities within.) The initial data for the first time step are given by

$$U(x,0)=\begin{cases} U^0_{k-2}, & x<k\triangle x,\\ U^0_k, & x>k\triangle x.\end{cases}$$

Here $\{U^0_k; k=0,\pm 2,\ldots\}$ is a set of constant states that approximate $U_0(x)$. To start we solve for the sequence of Riemann problems

$$U_t+F(U)_x=0,\qquad U(x,0)=\begin{cases} U^0_{k-2}, & x<k\triangle x,\\ U^0_k, & x>k\triangle x.\end{cases}\tag{19}$$

Now let $v^0(x,t)$ denote the solution obtained by solving the Riemann problems at the first time step. Glimm's idea is to choose initial data $U^1(x,t)$ at $t=\triangle t$ by random choice, and thereby pose Riemann problems at the next time step. This means that the initial data $U^1_k(x,\triangle t)$ for the Riemann problems at $t=\triangle t$ are chosen by

$$U^1_k(x,\triangle t)=v^0((k+1)\triangle x+\theta_1\triangle x,\triangle t)\quad \text{for } k\triangle x\leqslant x\leqslant (k+2)\triangle x.$$

Here θ_1 is a random number between 1 and -1. We repeat the process for each time step, and the initial condition for the ith time step will be

$$U^i_k(x,i\triangle t)=v^{i-1}((k+1)\triangle x+\theta_i\triangle x,i\triangle t)\quad \text{for } k\triangle x\leqslant x\leqslant (k+2)\triangle x,$$

where $\{\theta_i : i=1,2,\ldots\}$ is a set of random numbers between 1 and -1, $k+i+1\equiv 0\,(\mathrm{mod}\,2)$, and $v^{i-1}(x,t)$ is the solution given by solving the Riemann problems in the ith time step. Let $\{U_{\theta,\triangle x}\}$ denote the approximate solution with $\theta\equiv(\theta_1,\theta_2,\ldots)$. It is easy to see that $\{U_{\theta,\triangle x}\}$ depends on the choice of θ and the size of the grid $\triangle x$.

Next we describe the wave interactions. Here we use the notations in [20]. Let $(U_\mathrm{L},U_\mathrm{R})$ denote the solution of the Riemann problem consisting of constant states $U_\mathrm{L}\equiv U_0,U_1,\ldots,U_{n+1}\equiv U_\mathrm{R}$ with the parameterization $T^k_{\varepsilon_k}(U_k)=U_{k+1}$; then the solution $(U_\mathrm{L},U_\mathrm{R})$ can be written as

$$(U_\mathrm{L},U_\mathrm{R})\equiv((U_0,U_1,\ldots,U_{n+1})/(\varepsilon_0,\ldots,\varepsilon_n)).$$

We choose U_m as a constant state near U_L and U_R, and we can also write

$$(U_\mathrm{L},U_m)\equiv((\bar U_0,\bar U_1,\ldots,\bar U_{n+1})/(\gamma_0,\ldots,\gamma_n)),$$

$$(U_m,U_\mathrm{R})\equiv((\tilde U_0,\tilde U_1,\ldots,\tilde U_{n+1})/(\beta_0,\ldots,\beta_n)).$$

The parameter ε_i is called the *wave strength* of the *i-wave* that connects the states U_i and U_{i+1}. We have the following propositions for the relation of ε_i, γ_i and β_i, cf. [2,20]:

$$\varepsilon_i = \gamma_i + \beta_i + O(|\gamma||\beta|), \ \ i = 0, 1, \ldots, n. \tag{20}$$

We say that the *i-wave* and *j-wave* are *approaching waves* if either (i) the wave on the left belongs to the larger characteristic family or (ii) if both waves come from the same characteristic family, and at least one wave is a shock, cf. [20]. For the waves that are approaching, the following result holds:

$$\varepsilon_i = \gamma_i + \beta_i + O(1)D(\gamma, \beta) \quad \text{as } |\gamma| + |\beta| \to 0, \ \ i = 0, 1, \ldots, n. \tag{21}$$

Here $D(\gamma, \beta)=\sum |\gamma_i||\beta_j|$; the sum is over all pairs for which the *i-wave* and *j-wave* are approaching. We also define what are mesh points, mesh curves and immediate successors. The points $\{P_i\}$ are *mesh points* on the (x, t)-plane if the initial data are determined by the values of the solution at those points. We can connect those $\{P_i\}$ to obtain a set of the diamond regions. We call an unbounded piecewise linear curve I a *mesh curve* if I lies on the boundaries of those diamond regions. So if I is a mesh curve, then I divides the x–t plane into I^+ and I^- parts, such that I^- contains $t = 0$; we say two mesh curves $I_1 > I_2$ if every point of I_1 is either on I_2 or contained in I_2^+. And I_1 is an immediate successor of I_2 if $I_1 > I_2$ and every mesh point of I_1 except one is on I_2, cf. [20].

Given a mesh curve I. Define the Glimm functionals

$$Q(I) = \sum \{|\gamma||\beta| : \gamma, \beta \text{ cross } I \text{ and approach } \},$$
$$L(I) = \sum \{|\alpha| : \alpha \text{ crosses } I \ \},$$

where α, γ, β are waves in the approximate solution $\{U_{\theta,\triangle x}\}$. It is easy to see that the functional L is equivalent to the total variation of $\{U_{\theta,\triangle x}\}$.

Theorem 3.1 (*Smoller [20]*). *Let I and J be two mesh curves with $J > I$; suppose I is in the domain of $\{U_{\theta,\triangle x}\}$. If $L(I)$ is sufficiently small, then J is also in the domain of $\{U_{\theta,\triangle x}\}$. Furthermore, $Q(I) \geqslant Q(J)$ and there exists a constant k which is independent of J such that $L(I) + kQ(I) \geqslant L(J) + kQ(J)$. And if the total variation of $U_0(x)$ is small, then $\{U_{\theta,\triangle x}\}$ is defined for $t \geqslant 0$.*

Theorem 3.2 (*Smoller [20]*). *Let $TV(U)$ denote the total variation of U. If $TV(U_0)$ is small, then*

(i) $TV(U_{\theta,\triangle x}) \leqslant C_1 TV(U_0)$, *$C_1$ is independent of θ and $\triangle x$.*
(ii) $TV[U_{\theta,\triangle x}(x, n\triangle t)] + sup_x\, [U_{\theta,\triangle x}(x, n\triangle t)] < C_2 TV(U_0)$, *$C_2$ is independent of n, θ, $\triangle x$ and $\triangle t$.*
(iii) $\int_R \mid U_{\theta,\triangle x}(x, t) - U_{\theta,\triangle x}(x, \bar{t}) \mid dx \leqslant C_3(\mid t - \bar{t} \mid + \triangle t)$, *$C_3$ is independent of θ and $\triangle x$.*

We now discuss the convergence of the approximate solutions for both homogeneous and inhomogeneous cases. The proofs for both cases are essentially the same, because the structures of the wave curves for the solution of the Riemann problems in both cases are essentially the same. Thus the following convergence theorem applied in both cases.

Theorem 3.3. *Let $\{U_{\theta,\triangle x}\}$ be the approximate solution for the Cauchy problem* (3) (*or the homogeneous case* (18)) *generated by the Glimm scheme. Then there exists a subsequence $\{U_{\theta,\triangle x_i}\}$ of $\{U_{\theta,\triangle x}\}$ such that $\{U_{\theta,\triangle x_i}\}$ converges to some measurable function $U(x,t)$ in the L^1_{loc} sense.*

Next we review the argument that the limit function $U(x,t)$ of $\{U_{\theta,\triangle x}\}$ is a weak solution for almost any sample sequence θ. We restrict ourselves to homogeneous cases, where Glimm's argument applies directly. A weak solution $U(x,t)$ of the Cauchy problem (18), by definition, satisfies

$$\bar{R}_\phi(U) \equiv \int\int_{t>0} U\phi_t + F(U)\phi_x + \int_{-\infty}^{\infty} U_0(x)\phi(x,0)\,dx = 0, \quad \forall \phi \in C_0^1(R^2).$$

From the work of Glimm, we obtain that, for almost any choice of θ,

$$\begin{aligned}\lim_{\triangle x_i \to 0} \bar{R}_\phi(U_{\theta,\triangle x_i}) \equiv & \lim_{\triangle x_i \to 0}\left(\int\int_{t>0} (U_{\theta,\triangle x_i})\phi_t + F(U_{\theta,\triangle x_i})\phi_x\,dx\,dt\right) \\ & + \int_{-\infty}^{\infty} U_0(x)\phi(x,0)\,dx \\ = {}& 0 \qquad \text{for all } \phi \in C_0^1(R^2).\end{aligned}$$

Then we can use the Lebesgue bounded convergence theorem to pass the limit sign inside the $\bar{R}_\phi$ to get

$$\bar{R}_\phi\left(\lim_{\triangle x_i \to 0} U_{\theta,\triangle x_i}\right) = 0 \qquad \text{for all } \phi \in C_0^1(R^2).$$

By the smoothness of $F(U)$ and the Lebesgue bounded convergence theorem, we have

$$F(U_{\theta,\triangle x_i}) \longrightarrow F(U(x,t)) \text{ in } L_1^{\mathrm{loc}},$$

where $U(x,t)$ is the measurable function described in Theorem 3.3. So we have the following existence theorem for the weak solution of the Cauchy problem (18):

Theorem 3.4 (*Glimm [2]*). *Consider the Cauchy problem* (18) *for a strictly hyperbolic system in a neighborhood of state $\tilde{U}$. Assume that the initial data $U_0(x)$ have a* *sufficiently small total variation. Then there is a weak solution* $U(x,t)$ *of* (18) *defined for all* x *and all* $t \geqslant 0$, *such that*

$$\|U - \tilde{U}\|_\infty \leqslant K \cdot \|U(\cdot, 0) - \tilde{U}\|_\infty, \tag{22}$$

$$TV\{U(\cdot, t)\} \leqslant K(TV\{U(\cdot, 0)\}), \tag{23}$$

$$\int_R |\, U(x, t_1) - U(x, t_2) \,|\; dx \leqslant K \,|\, t_1 - t_2 \,|\, (TV\{U(\cdot, 0)\}) \tag{24}$$

for some constant K.

In the inhomogeneous case, we must modify Glimm's argument in order to prove the existence theorem. By Theorem 3.2, we already have conditions (22) to (24), but it remains to give conditions under which the limit function $U(x,t)$ is a weak solution of the Cauchy problem (3) in the inhomogeneous case. The proof will be given in the next section.

For the inhomogeneous case, the function $U(x,t)$ that the $\{U_{\theta,\triangle x_i}\}$ converges to in L^1_{loc} is also a measurable function. To show that $U(x,t)$ is a weak solution of the Cauchy problem (3), we require the value of the *Residual* $R_\phi(U(x,t))$ to be 0. From the experience with the homogeneous case, to calculate $R_\phi(U(x,t))$, we need to compute $R_\phi(U_{\theta,\triangle x})$. But the term $\int\int_{t>0} a'_{\theta,\triangle x} G(U_{\theta,\triangle x}, x)\phi \, dx\, dt$ in $R_\phi(U_{\theta,\triangle x})$ contains the product of the *delta* function $a'_{\theta,\triangle x}$ and discontinuous function $G(U_{\theta,\triangle x})$, which means that $\int\int_{t>0} a' G(U_{\theta,\triangle x}, x)\phi \, dx\, dt$ has no meaning in the distribution sense. In order to overcome this difficulty, we reconstruct the approximate solution $\{U_{\theta,\triangle x}\}$ to give meaning to $\int\int_{t>0} a'_{\theta,\triangle x} G(U_{\theta,\triangle x}, x)\phi \, dx\, dt$ as a distribution. In the next section, we will describe the reconstruction of $\{U_{\theta,\triangle x}\}$ and obtain the existence theorem for the inhomogeneous case.

4. Weak convergence and existence theorem for the inhomogeneous cases

In this section, we construct the modified approximate solution by the generalized Glimm scheme and derive the convergence theorem for the inhomogeneous case. In the end we prove the main theorem of the paper, the existence theorem for the Cauchy problem (3).

In Section 3, we already mentioned the construction of the approximate solution $\{U_{\theta,\triangle x}\}$ by the Glimm scheme, and it causes the non-integrability of $\int\int_{t>0} a'_{\theta,\triangle x} G(U_{\theta,\triangle x}, x)\phi \, dx\, dt$ in the *Residual*

$$R_\phi(U) = \int\int_{t>0} U\phi_t + F(U)\phi_x + a' G(U, x)\phi \, dx\, dt + \int_{-\infty}^{\infty} U_0(x)\phi(x, 0)\, dx$$

for $\phi \in C_0^1(R^2)$. In order to let $R_\phi(U)$ be well defined in the distribution sense, we need to smooth out the standing wave discontinuities in the $\{U_{\theta,\triangle x}\}$.

First we describe the new (modified) approximate solution which is dependent on the original approximate solution by the Glimm scheme. We know that for $\{U_{\theta,\triangle x}\}$, the standing wave discontinuities on each ith time step are located at each $k\triangle x$ for $k+i=0\,(\mathrm{mod}\,2)$. In the new approximate solution $\{U^{\varepsilon}_{\theta,\triangle x}; 0<\varepsilon<1\}$, we replace the standing wave discontinuities by smooth standing waves located between $(k-\varepsilon)\triangle x$ and $(k+\varepsilon)\triangle x$ and solve the set of initial value problems of the ordinary differential equations described in Section 2,

$$DF(U^{\varepsilon}(x))\cdot \dot{U}^{\varepsilon}(x)=G(U^{\varepsilon}(x),x),$$

$$U_0^{\varepsilon}(x)=\begin{cases} U_{\mathrm{L}}^k, & x<(k-\varepsilon)\triangle x,\\ (a_{\mathrm{s}}^{\varepsilon}(x),u_{1\mathrm{s}}(a_{\mathrm{s}}^{\varepsilon}(x)),\ldots,u_{n\mathrm{s}}(a_{\mathrm{s}}^{\varepsilon}(x))), & (k-\varepsilon)\triangle x\leqslant x\leqslant(k+\varepsilon)\triangle x,\\ U_{\mathrm{R}}^k, & x>(k+\varepsilon)\triangle x,\end{cases}$$

on the ith time step, $k+i=0\,(\mathrm{mod}\,2)$. The set $\{U_{\mathrm{L}}^k, U_{\mathrm{R}}^k; k=\cdot,-1,0,1,\cdot\}$ are the constant states connected by the standing wave discontinuities on the ith time step. So in the region $[(k-\varepsilon)\triangle x,(k+\varepsilon)\triangle x]\times[i\triangle t,(i+1)\triangle t]$, $\{U^{\varepsilon}_{\theta,\triangle x}\}$ has a smooth standing wave $(a_{\mathrm{s}}^{\varepsilon}(x),u_{1\mathrm{s}}(a_{\mathrm{s}}^{\varepsilon}(x)),\ldots,u_{n\mathrm{s}}(a_{\mathrm{s}}^{\varepsilon}(x)))$ between $(k-\varepsilon)\triangle x$ and $(k+\varepsilon)\triangle x$, and agrees with the solution solved by the homogeneous case in the rest of the region (except that the positive speed waves are shifted ε units to the right, and the negative speed waves are shifted ε units to the left). The new approximate solution $\{U^{\varepsilon}_{\theta,\triangle x}\}$ depends on the width of the interval that the standing wave discontinuities had been smoothed out, so it also depends on ε. And for the random number θ, since $\{U^{\varepsilon}_{\theta,\triangle x}\}$ is constructed after we use the Glimm scheme, θ has the same effect on both $\{U_{\theta,\triangle x}\}$ and $\{U^{\varepsilon}_{\theta,\triangle x}\}$. We have Fig. 4 for $\{U_{\theta,\triangle x}\}$ and $\{U^{\varepsilon}_{\theta,\triangle x}\}$ in the region $[(k-1)\triangle x,(k+1)\triangle x]\times[i\triangle t,(i+1)\triangle t]$.

Next we describe some properties of $\{U^{\varepsilon}_{\theta,\triangle x}\}$. First, our choice of $\phi_{\varepsilon}(x)$ described in (15) of Section 2 implies that $a_{\mathrm{s}}^{\varepsilon}(x)$ in $U_{\mathrm{s}}^{\varepsilon}(x)$ is a monotone function of x. And by the assumption that the n-vector

$$\left(\frac{\partial f}{\partial u}\right)^{-1}\left(g-\frac{\partial f}{\partial a}\right)(U)\neq 0 \quad \text{for } U\in\Omega, \tag{25}$$

$\{u_{i\mathrm{s}}(a_{\mathrm{s}}^{\varepsilon}(x));\ i=1,\ldots,n\}$ are monotone functions of x. So it is easy to see that $\{U^{\varepsilon}_{\theta,\triangle x}\}$ is uniformly bounded since $\{U_{\theta,\triangle x}\}$ is uniformly bounded. And the oscillation of $\{U^{\varepsilon}_{\theta,\triangle x}\}$ in $[(k-1)\triangle x,(k+1)\triangle x]\times[i\triangle t,(i+1)\triangle t]$ will be equal to the oscillation of $\{U_{\theta,\triangle x}\}$ for any i,k and $0<\varepsilon<1$ since $\{U^{\varepsilon}_{\theta,\triangle x}\}$ is a monotone function of x.

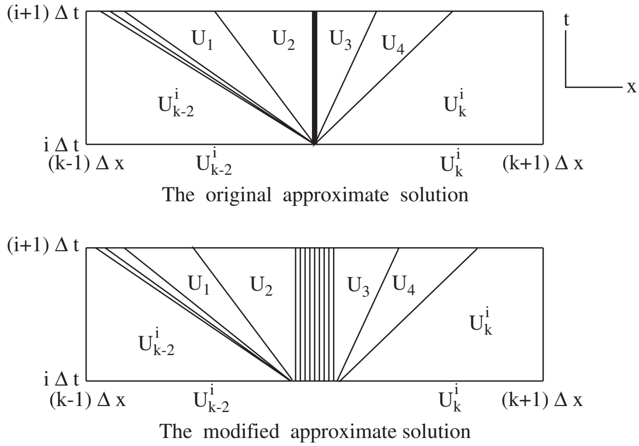

Fig. 4. The original and modified approximate solutions of the Cauchy problem.

Recall the result in Section 2, there are infinitely many choices of $a_s^\varepsilon(x)$ for $U_s^\varepsilon(x)$, which means that we have infinitely many choices of $\{U^\varepsilon_{\theta,\triangle x}\}$. By the monotonicity of $a_s^\varepsilon(x)$ and $\{u_{is}(a_s^\varepsilon(x));\ i=1,\dots,n\}$ we see that all the $\{U_s^\varepsilon(x)\}$ converge pointwise to the same $U_s(x)$ in $\{U_{\theta,\triangle x}\}$ as ε approaches 0 since all the $\{a_s^\varepsilon(x)\}$ we choose converge to $a_s(x)$ of $\{U_s(x)\}$. We will use these properties of $\{U^\varepsilon_{\theta,\triangle x}\}$ to prove the convergence theorem.

In Section 3, we already described the convergence of the $\{U_{\theta,\triangle x}\}$. Here, we use Helly's theorem again to prove the convergence of $\{U^\varepsilon_{\theta,\triangle x}\}$. First we prove the following theorem which is similar to Theorem 3.2.

Theorem 4.1. *Let $\{U^\varepsilon_{\theta,\triangle x}\}$ be the modified approximate solution, and $U_s^\varepsilon(x)\equiv(a_s^\varepsilon(x), u_{1s}(a_s^\varepsilon(x)),\dots,u_{ns}(a_s^\varepsilon(x)))$ be the collection of smooth standing wave solutions in $\{U^\varepsilon_{\theta,\triangle x}\}$. If the total variation of $U_0(x)$ is small, then given any ε, $0<\varepsilon<1$, and monotone function $\{a_s^\varepsilon(x)\}$ of x, we have*

(I) $TV(U^\varepsilon_{\theta,\triangle x})\leqslant K_1\ TV(U_0)$, *$K_1$ is independent of θ and $\triangle x$.*

(II) $TV[U^\varepsilon_{\theta,\triangle x}(x,n\triangle t)]+sup_x\,[U^\varepsilon_{\theta,\triangle x}(x,n\triangle t)]<K_2\ TV(U_0)$, *$K_2$ is independent of n, θ, $\triangle x$ and $\triangle t$.*

(III) $\int_R \mid U^\varepsilon_{\theta,\triangle x}(x,t)-U^\varepsilon_{\theta,\triangle x}(x,\bar t)\mid dx\leqslant K_3(\mid t-\bar t\mid+\triangle t)$, *$K_3$ is independent of θ and $\triangle x$.*

Proof. We start from part (I). By the monotonicity of $U_s^\varepsilon(x)$ and Theorem 3.2. we know that

$$TV(U^\varepsilon_{\theta,\triangle x})=TV(U_{\theta,\triangle x})\leqslant K_1 TV(U_0)$$

for some constant K_1. For (II), by the construction of $\{U^{\varepsilon}_{\theta,\triangle x}\}$ and the monotonicity of $U^{\varepsilon}_{s}(x)$, we have $sup_x[U^{\varepsilon}_{\theta,\triangle x}(x, n\triangle t)] = sup_x\,[U_{\theta,\triangle x}(x, n\triangle t)]$. So (II) holds directly by (I). For (III), we have

$$\begin{aligned}\int_R \mid U^{\varepsilon}_{\theta,\triangle x}(x,t) - U^{\varepsilon}_{\theta,\triangle x}(x,\bar{t}) \mid \, dx \leqslant & \int_R \mid U^{\varepsilon}_{\theta,\triangle x}(x,t) - U_{\theta,\triangle x}(x,t) \mid \, dx \\ & + \int_R \mid U_{\theta,\triangle x}(x,t) - U_{\theta,\triangle x}(x,\bar{t}) \mid \, dx \\ & + \int_R \mid U_{\theta,\triangle x}(x,\bar{t}) - U^{\varepsilon}_{\theta,\triangle x}(x,\bar{t}) \mid \, dx.\end{aligned}$$

Let I^1, I^2 and I^3 denote these three terms on the right-hand side of the inequality in order. We have

$$I^2 \leqslant C_3 \cdot (\mid t - \bar{t} \mid + \triangle t)$$

for some constant C_3 by Theorem 3.2. To estimate I^1, we have

$$\begin{aligned}I^1 &\equiv \int_R \mid U^{\varepsilon}_{\theta,\triangle x}(x,t) - U_{\theta,\triangle x}(x,t) \mid \, dx \\ &= \sum_{m=-\infty}^{\infty} \int_{(m-1)\triangle x}^{(m+1)\triangle x} \mid U^{\varepsilon}_{\theta,\triangle x}(x,t) - U_{\theta,\triangle x}(x,t) \mid \, dx \\ &\leqslant \sum_{m=-\infty}^{\infty} \int_{(m-1)\triangle x}^{(m+1)\triangle x} 2\text{osc.}(U_{\theta,\triangle x})\, dx \\ &\leqslant Const. \cdot TV(U_{\theta,\triangle x}) \cdot \triangle x \\ &\leqslant Const. \cdot TV(U_0(x)) \cdot \triangle x \\ &\leqslant Const. \cdot \triangle x \\ &\leqslant C_1 \cdot \triangle t.\end{aligned}$$

Here osc.$(U_{\theta,\triangle x})$ is the oscillation of $U_{\theta,\triangle x}$. The constant C_1 is independent of θ and $\triangle x$. Similarly, we have $I^3 \leqslant C_2 \triangle t$ for some constant C_2 independent of θ and $\triangle x$. Choose K_3 as

$$K_3 = 2\, sup(C_1, C_2, C_3).$$

We have proved the results of (III). $\square$

The following convergence theory is just an extension of Theorem 3.3.

Theorem 4.2. *Let $\{U^{\varepsilon}_{\theta,\triangle x}\}$ be a modified approximate solution for the Cauchy problem* (3), *and let $U^{\varepsilon}_{\mathrm{s}}(x) \equiv (a^{\varepsilon}_{\mathrm{s}}(x), u_{1\mathrm{s}}(a^{\varepsilon}_{\mathrm{s}}(x)), \ldots, u_{n\mathrm{s}}(a^{\varepsilon}_{\mathrm{s}}(x)))$ be a collection of smooth standing wave solutions in $\{U^{\varepsilon}_{\theta,\triangle x}\}$. If the total variation of $U_0(x)$ is small, then given any ε, $0<\varepsilon<1$, and monotone function $\{a^{\varepsilon}_{\mathrm{s}}(x)\}$ of x, we can find a subsequence the $\{U^{\varepsilon}_{\theta,\triangle x_i}\}$ of $\{U^{\varepsilon}_{\theta,\triangle x}\}$ such that $\{U^{\varepsilon}_{\theta,\triangle x_i}\}$ converges to some function $U^{\varepsilon}(x,t)$ in the L^1_{loc} sense. And there exists a measurable function $U(x,t)$, Lipschitz-continuous in a, such that $U^{\varepsilon}(x,t) \longrightarrow U(x,t)$ in L^1_{loc} as $\varepsilon \to 0$.*

Theorem 4.3. *Suppose we have the same assumption as in Theorem* 4.2. *Then given any ε, $0<\varepsilon<1$, there exists a subsequence $\{U^{\varepsilon}_{\theta,\triangle x_i}\}$ such that $F(U^{\varepsilon}_{\theta,\triangle x_i}) \longrightarrow F(U^{\varepsilon}(x,t))$ in L^1_{loc} for every continuous function F. Furthermore, $F(U^{\varepsilon}(x,t)) \longrightarrow F(U(x,t))$ in L^1_{loc} for every continuous function F as $\varepsilon \to 0$. The function $U(x,t)$ is as in Theorem* 4.2.

The proofs of Theorems 4.2 and 4.3 are based on Theorem 4.1, Helly's selection principle and the Lebesgue bounded convergence theorem.

Now the term $\int\int_{t>0} a'_{\varepsilon}G(U^{\varepsilon}_{\theta,\triangle x}, x)\phi\,dx\,dt$ in $R_{\phi}(U^{\varepsilon}_{\theta,\triangle x})$ becomes integrable, so we are able to calculate $R_{\phi}(U^{\varepsilon}_{\theta,\triangle x})$ to show that $R_{\phi}(U^{\varepsilon}_{\theta,\triangle x})$ vanishes as ε, $\triangle x$ approach 0. We will also study the compactness of $a'_{\varepsilon}G(U^{\varepsilon}_{\theta,\triangle x_i})$. And it leads to the existence of the weak solution for Cauchy problem (3), which is the main goal of the paper.

First we define the residual $R_{\phi}(U)$ for the Cauchy problem (3),

$$R_{\phi}(U) \equiv \int\int_{t>0} U\phi_t + F(U)\phi_x + a'G(U,x)\phi\,dx\,dt + \int_{-\infty}^{\infty} U_0(x)\phi(x,0)\,dx \tag{26}$$

for all $\phi \in C^1_0(R^2)$. Then by the definition of the weak solution for Cauchy problem (3), we have

$$U(x,t) \text{ is a weak solution for Cauchy problem (3)} \quad \text{iff} \quad R_{\phi}(U)=0 \tag{27}$$

for all $\phi \in C^1_0(R^2)$.

Let $\{U_{\theta,\triangle x}\}$ and $\{U^{\varepsilon}_{\theta,\triangle x}\}$ be the approximate solution and modified approximate solution; also let E_i denote the ith time strip. Since $U^{\varepsilon}_{\theta,\triangle x}$ is a weak solution for each E_i, by the divergence theorem on each E_i, we obtain

$$\begin{aligned} R_{\phi}(U^{\varepsilon}_{\theta,\triangle x}) = {} & \sum_i \int\int_{E_i} U^{\varepsilon}_{\theta,\triangle x}\phi_t + F(U^{\varepsilon}_{\theta,\triangle x})\phi_x + a'_{\varepsilon}G(U^{\varepsilon}_{\theta,\triangle x}, x)\phi\,dx\,dt \\ & + \int_{-\infty}^{\infty} U_0(x)\phi(x,0)\,dx \end{aligned}$$

$$= -\sum_i \int\int_{E_i} (U^\varepsilon_{\theta,\triangle x})_t \phi + ((F(U^\varepsilon_{\theta,\triangle x}))_x)\phi - a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x}, x)\phi \, dx\, dt$$

$$+ \sum_i \int_{\partial E_i} (U^\varepsilon_{\theta,\triangle x}\phi) \cdot n_t \, \mathrm{d}S + \sum_i \int_{\partial E_i} (F(U^\varepsilon_{\theta,\triangle x})\phi) \cdot n_x \, \mathrm{d}S$$

$$+ \int_{-\infty}^{\infty} U_0(x)\phi(x,0)\, dx.$$

Here ∂E_i is the boundary of E_i with the outer normal (n_x, n_t), and $\phi \in C^1_0(R^2)$. It is easy to see that

$$\sum_i \int\int_{E_i} (U^\varepsilon_{\theta,\triangle x})_t \phi + ((F(U^\varepsilon_{\theta,\triangle x}))_x)\phi - a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x}, x)\phi \, dx\, dt = 0,$$

$$\sum_i \int_{\partial E_i} (F(U^\varepsilon_{\theta,\triangle x})\phi) \cdot n_x \, dS = 0.$$

So it follows that

$$\begin{aligned}
R_\phi(U^\varepsilon_{\theta,\triangle x}) &= \sum_i \int_{\partial E_i} (U^\varepsilon_{\theta,\triangle x}\phi) \cdot n_t \, dS + \int_{-\infty}^{\infty} U_0(x)\phi(x,0)\, dx \\
&= \sum_i \int_{-\infty}^{\infty} [U^\varepsilon_{\theta,\triangle x}](x, i\triangle t)\phi(x, i\triangle t)\, dx \\
&\quad + \int_{-\infty}^{\infty} (U^\varepsilon_{\theta,\triangle x}(x,0) - U_0(x))\phi(x,0)\, dx \\
&\equiv \sum_i J^i_\varepsilon(\theta, \triangle x, \phi) + \int_{-\infty}^{\infty} (U^\varepsilon_{\theta,\triangle x}(x,0) - U_0(x))\phi(x,0)\, dx \\
&\equiv J_\varepsilon(\theta, \triangle x, \phi) + \int_{-\infty}^{\infty} (U^\varepsilon_{\theta,\triangle x}(x,0) - U_0(x))\phi(x,0)\, dx,
\end{aligned}$$

where

$$\begin{aligned}
[U^\varepsilon_{\theta,\triangle x}](x, i\triangle t) &\equiv U^\varepsilon_{\theta,\triangle x}(x, i\triangle t^+) - U^\varepsilon_{\theta,\triangle x}(x, i\triangle t^-), \\
J^i_\varepsilon(\theta, \triangle x, \phi) &\equiv \int_{-\infty}^{\infty} [U^\varepsilon_{\theta,\triangle x}](x, i\triangle t)\phi(x, i\triangle t)\, dx, \\
J_\varepsilon(\theta, \triangle x, \phi) &\equiv \sum_i J^i_\varepsilon(\theta, \triangle x, \phi).
\end{aligned}$$

First we estimate the term $\int_{-\infty}^{\infty}(U^{\varepsilon}_{\theta,\triangle x}(x,0)-U_0(x))\phi(x,0)\,dx$.

$$
\begin{aligned}
&\left|\int_{-\infty}^{\infty}(U^{\varepsilon}_{\theta,\triangle x}(x,0)-U_0(x))\phi(x,0)\,dx\right| \\
&\leqslant \int_{-\infty}^{\infty} \mid (U^{\varepsilon}_{\theta,\triangle x}(x,0)-U_0(x)) \mid\mid \phi(x,0) \mid \; dx \\
&\leqslant \parallel \phi \parallel_{\infty} \int_{-\infty}^{\infty} \mid (U^{\varepsilon}_{\theta,\triangle x}(x,0)-U_{\theta,\triangle x}(x,0) \mid + \mid (U_{\theta,\triangle x}(x,0)-U_0(x) \mid \; dx \\
&\leqslant K_1 \parallel \phi \parallel_{\infty} \cdot (TV\{U_{\theta,\triangle x}\})(2\triangle x) + K_2 \parallel \phi \parallel_{\infty} \cdot (TV\{U_0(x)\})(2\triangle x) \\
&\leqslant Const \cdot TV\{U_0(x)\}(2\triangle x),
\end{aligned}
$$

where $\parallel \phi \parallel_{\infty}=$ the sup-norm of ϕ. From previous estimation, if $TV\{U_0(x)\}$ is small, then

$$
\int_{-\infty}^{\infty}(U^{\varepsilon}_{\theta,\triangle x}(x,0)-U_0(x))\phi(x,0)\,dx \;\; \to 0 \text{ as } \triangle x \to 0.
$$

Next we have the following estimations for $J^i_{\varepsilon}(\theta,\triangle x,\phi)$ and $J_{\varepsilon}(\theta,\triangle x,\phi)$.

Theorem 4.4.

$$
\begin{aligned}
|J^i_{\varepsilon}(\theta,\Delta x,\phi)| &\leqslant K\|\phi\|_{\infty}(\Delta x) \qquad \textit{for some } K, \\
|J_{\varepsilon}(\theta,\Delta x,\phi)| &\leqslant \bar{K}(r(\phi))\|\phi\|_{\infty} \quad \textit{for some constant } \bar{K},
\end{aligned}
$$

where $r(\phi) =$ diameter of support(ϕ).

Proof. We break the proof into two cases.

(i) Assume i is even. In this case, the modified approximate solution $\{U^{\varepsilon}_{\theta,\Delta x}\}$ in $I_m=[(m-1)\Delta x,(m+1)\Delta x]$ of E_i *and* E_{i-1} is described in Fig. 5.

First we define the following intervals:

$$
\begin{aligned}
I_1 &= [(m-1)\Delta x,(m-1+\varepsilon)\Delta x], \\
I_2 &= [(m-\varepsilon)\Delta x,(m+\varepsilon)\Delta x], \\
I_3 &= [(m+1-\varepsilon)\Delta x,(m+1)\Delta x], \\
I_m &= [(m-1)\Delta x,(m+1)\Delta x].
\end{aligned}
$$

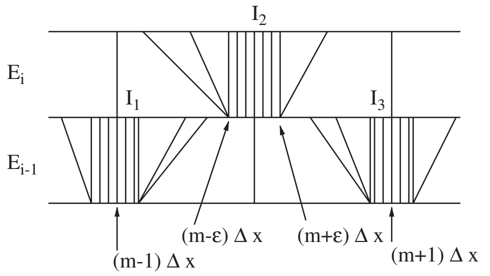

Fig. 5. The case for $i =$ even number.

We have

$$\begin{aligned}
&|J_\varepsilon^i(\theta, \Delta x, \phi)| \\
&\quad = \left| \int_{-\infty}^{\infty} [U_{\theta,\Delta x}^\varepsilon]_i \phi(x, i\Delta t)\, dx \right| \\
&\quad \leqslant \|\phi\|_\infty \left| \int_{-\infty}^{\infty} [U_{\theta,\Delta x}^\varepsilon]_i\, dx \right| \\
&\quad \leqslant \|\phi\|_\infty \sum_{m=-\infty}^{\infty} \int_{(m-1)\Delta x}^{(m+1)\Delta x} |U_{\theta,\Delta x}^\varepsilon(x, t_i^+) - U_{\theta,\Delta x}^\varepsilon(x, t_i^-)|\, dx. \\
&\quad \leqslant \|\phi\|_\infty \sum_{m=-\infty}^{\infty} \left(\int_{I_1} + \int_{I_2} + \int_{I_3} + \int_{I_m/I_1 \cup I_2 \cup I_3} \right) \\
&\qquad \times |U_{\theta,\Delta x}^\varepsilon(x, t_i^+) - U_{\theta,\Delta x}^\varepsilon(x, t_i^-)|\, dx \\
&\quad \equiv \|\phi\|_\infty \sum_m (S_{m1} + S_{m2} + S_{m3} + S_{m4}).
\end{aligned}$$

To obtain the estimation of S_{m1}, it follows directly by the construction of $U_{\theta,\Delta x}$, we have

$$\begin{aligned}
S_{m1} &\equiv \int_{I_1} |U_{\theta,\Delta x}^\varepsilon(x, t_i^+) - U_{\theta,\Delta x}^\varepsilon(x, t_i^-)|\, dx \\
&= \int_{I_1} |U_{\theta,\Delta x}(m\Delta x + \theta_i \Delta x, t_i^-) - U_{\theta,\Delta x}^\varepsilon(x, t_i^-)|\, dx \\
&\leqslant \int_{I_1} |U_{\theta,\Delta x}(m\Delta x + \theta_i \Delta x, t_i^-) - U_{\theta,\Delta x}(x, t_i^-)|\, dx
\end{aligned}$$

$$
\begin{aligned}
&+\int_{I_1} |U_{\theta,\Delta x}(x,t_i^-) - U^{\varepsilon}_{\theta,\Delta x}(x,t_i^-)|\,dx \\
&\leqslant Const \cdot \{TV \text{ of } U_{\theta,\Delta x} \text{ in } I_m \text{ on } E_{i-1}\} \cdot (\varepsilon\triangle x).
\end{aligned}
$$

Similarly,

$$
S_{m3} \leqslant Const \cdot \{TV \text{ of } U_{\theta,\Delta x} \text{ in } I_m \text{ on } E_{i-1}\} \cdot (\varepsilon\triangle x).
$$

For S_{m2}, we see that

$$
\begin{aligned}
S_{m2} &\equiv \int_{I_2} \mid U^{\varepsilon}_{\theta,\Delta x}(x,t_i^+) - U^{\varepsilon}_{\theta,\Delta x}(x,t_i^-) \mid \,dx \\
&\leqslant \int_{I_2} \mid U^{\varepsilon}_{\theta,\Delta x}(x,t_i^+) - U_{\theta,\Delta x}(x,t_i^+) \mid \,dx \\
&\quad + \int_{I_2} \mid U_{\theta,\Delta x}(x,t_i^+) - U_{\theta,\Delta x}(x,t_i^-) \mid \,dx \\
&\quad + \int_{I_2} \mid U_{\theta,\Delta x}(x,t_i^-) - U^{\varepsilon}_{\theta,\Delta x}(x,t_i^-) \mid \,dx \\
&\leqslant Const \cdot \{TV \text{ of } U_{\theta,\Delta x} \text{ in } I_m \text{ on } E_i\} \cdot (2\varepsilon\triangle x) \\
&\quad + Const \cdot \{TV \text{ of } U_{\theta,\Delta x} \text{ in } I_m \text{ on } E_{i-1}\} \cdot (2\varepsilon\triangle x).
\end{aligned}
$$

To estimate S_{m4}, we consider the following two cases:

(i) If x is on the left branch of $I_m / \cup_{j=1}^{3} I_j$, then case,

$$
U^{\varepsilon}_{\theta,\Delta x}(x,t_i^+) = U_{\theta,\Delta x}(m\triangle x + \theta_i\triangle x + \varepsilon\triangle x, t_i^-).
$$

(ii) If x is on the right branch of $I_m / \cup_{j=1}^{3} I_j$, then

$$
U^{\varepsilon}_{\theta,\Delta x}(x,t_i^+) = U_{\theta,\Delta x}(m\triangle x + \theta_i\triangle x - \varepsilon\triangle x, t_i^-).
$$

In both cases we obtain

$$
S_{m4} \leqslant \{TV \text{ of } U_{\theta,\Delta x} \text{ in } I_m \text{ on } E_{i-1}\} \cdot (2\triangle x).
$$

Thus,

$$
\begin{aligned}
S_{m1} + S_{m2} + S_{m3} + S_{m4} &\leqslant Const \cdot \{TV \text{ of } U_{\theta,\Delta x} \text{ in } I_m \text{ on } E_i\} \cdot (\triangle x) \\
&\quad + Const \cdot \{TV \text{ of } U_{\theta,\Delta x} \text{ in } I_m \text{ on } E_{i-1}\} \cdot (\triangle x),
\end{aligned}
$$

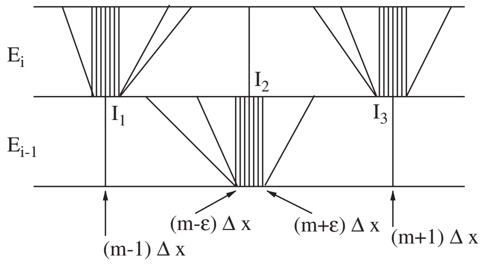

Fig. 6. The case for $i =$ odd number.

and this implies that

$$\begin{aligned}
|J_\varepsilon^i(\theta, \Delta x, \phi)| &\leqslant Const\cdot \parallel \phi \parallel_\infty TV\{U_{\theta,\triangle x} \text{ on } E_i\} \cdot (\triangle x) \\
&\quad + Const\cdot \parallel \phi \parallel_\infty \; TV\{U_{\theta,\Delta x} \text{ on } E_{i-1}\} \cdot (\triangle x) \\
&\leqslant Const\cdot \parallel \phi \parallel_\infty TV\{U_0(x)\} \cdot (\triangle x) \\
&\leqslant Const\cdot \parallel \phi \parallel_\infty \cdot (\triangle x).
\end{aligned}$$

We just proved

$$|\ J_\varepsilon^i(\theta, \Delta x, \phi)\ | \leqslant C_1 \parallel \phi \parallel_\infty \cdot (\triangle x)$$

for some constant C_1 when i is an even number.

(ii) Assume i is odd. See Fig. 6.

By the same calculation as in case (i), we have

$$|\ J_\varepsilon^i(\theta, \Delta x, \phi)\ | \leqslant C_2 \parallel \phi \parallel_\infty \cdot (\triangle x)$$

for some constant C_2 when i is an odd number. Now we choose $K = \max(C_1, C_2)$, we obtain the first estimation.

For $J_\varepsilon(\theta, \Delta x, \phi)$, we have

$$\begin{aligned}
|J_\varepsilon(\theta, \Delta x, \phi)| &= \left| \sum_{i\in\Lambda} J_\varepsilon^i(\theta, \Delta x, \phi) \right| \\
&\leqslant \sum_{i\in\Lambda} |J_\varepsilon^i(\theta, \Delta x, \phi)| \\
&\leqslant \sum_{i\in\Lambda} K\|\phi\|_\infty \cdot (\triangle x).
\end{aligned}$$

Here $|\Lambda|$ = number of $\{i;\ J^i_\varepsilon(\theta, \Delta x, \phi)$ is in the domain of *support*$(\phi)\}$, so $|\Lambda| = O(1)\cdot r(\phi)\cdot(\triangle t)^{-1}$. Plugging $|\Lambda|$ into the previous inequality, we have

$$\begin{aligned}|J_\varepsilon(\theta, \Delta x, \phi)| &\leqslant Const\cdot \parallel \phi \parallel_\infty r(\phi)\left(\frac{\triangle x}{\triangle t}\right),\\ &\leqslant Const\cdot sup \mid \lambda_i(U)\mid \cdot \parallel \phi \parallel_\infty \cdot r(\phi),\\ &\leqslant \bar{K}\cdot \parallel \phi \parallel_\infty \cdot r(\phi),\end{aligned}$$

for some constant $\bar{K}$. We prove the second estimation, and complete the proof. □

Next, following by Glimm's method, given a function ϕ with compact support and piecewise constant between $(m-1)\Delta x$ and $(m+1)\Delta x$ for each time step E_i, $m+i$ is even, we need to estimate

$$\langle J^i_\varepsilon(\theta, \Delta x, \phi), J^j_\varepsilon(\theta, \Delta x, \phi)\rangle \quad \text{for } i \neq j.$$

Here $\langle, \rangle$ is the L^2 product w.r.t. $\theta \in \Phi$. First we denote

$$J^i(\theta, \Delta x, \phi) \equiv \int_{-\infty}^{\infty} [U_{\theta,\triangle x}](x, i\triangle t)\phi(x, i\triangle t)\, dx.$$

From the result of Glimm [2], we have

$$\langle J^i(\theta, \Delta x, \phi), J^j(\theta, \Delta x, \phi)\rangle = 0 \text{ for } i \neq j.$$

And for $i < j$,

$$\begin{aligned}\langle J^i_\varepsilon(\theta, \Delta x, \phi), J^j(\theta, \Delta x, \phi)\rangle &\equiv \int\left(\int J^i_\varepsilon(\theta, \Delta x, \phi) J^j(\theta, \Delta x, \phi)\, d\theta_j\right)\prod_{l\neq j} d\theta_l\\ &= \int J^i_\varepsilon(\theta, \Delta x, \phi)\left(\int J^j(\theta, \Delta x, \phi)\, d\theta_j\right)\prod_{l\neq j} d\theta_l\\ &= 0,\end{aligned}$$

since $\int_{\theta_j\in\Phi} J^j(\theta, \Delta x, \phi)\, d\theta_j = 0$, which is from the result in [2,20].

Therefore,

$$\begin{aligned}&\langle J^i_\varepsilon(\theta, \Delta x, \phi), J^j_\varepsilon(\theta, \Delta x, \phi)\rangle\\ &\quad = \langle J^i_\varepsilon(\theta, \Delta x, \phi), J^j_\varepsilon(\theta, \Delta x, \phi)\rangle - \langle J^i_\varepsilon(\theta, \Delta x, \phi), J^j(\theta, \Delta x, \phi)\rangle\\ &\quad = \langle J^i_\varepsilon(\theta, \Delta x, \phi), J^j_\varepsilon(\theta, \Delta x, \phi) - J^j(\theta, \Delta x, \phi)\rangle\\ &\quad \leqslant \|J^i_\varepsilon(\theta, \Delta x, \phi)\|_\infty \cdot \|J^j_\varepsilon(\theta, \Delta x, \phi) - J^j(\theta, \Delta x, \phi)\|_{L^1}.\\ &\quad \leqslant K\|\phi\|_\infty(\Delta x)\cdot \|J^j_\varepsilon(\theta, \Delta x, \phi) - J^j(\theta, \Delta x, \phi)\|_{L^1}\end{aligned}$$

for some constant K. The last inequality is from Theorem 4.4.

Next we estimate the term $\|J_\varepsilon^j(\theta, \Delta x, \phi) - J^j(\theta, \Delta x, \phi)\|_{L^1}$.

$$\begin{aligned}\|J_\varepsilon^j(\theta, \Delta x, \phi) - J^j(\theta, \Delta x, \phi)\|_{L^1} &= \int_\Phi \left(\int_{\theta_j \in \Phi} (J_\varepsilon^j - J^j)\ d\theta_j \right) \prod_{l \neq i,j} d\theta_l \\ &\leqslant \int_\Phi \left(\int_{\theta_j \in \Phi} |J_\varepsilon^j - J^j|\ d\theta_j \right) \prod_{l \neq i,j} d\theta_l.\end{aligned}$$

Using the definitions of J_ε^j and J^j, we have

$$\begin{aligned}|J_\varepsilon^j - J^j| &= \int_{-\infty}^{\infty} |[U_{\theta,\Delta x}^\varepsilon] - [U_{\theta,\Delta x}]||\phi(x, i\Delta t)|\, dx \\ &\leqslant \|\phi\|_\infty \sum_{m=-\infty}^{\infty} \int_{(m-1)\Delta x}^{(m+1)\Delta x} |[U_{\theta,\Delta x}^\varepsilon] - [U_{\theta,\Delta x}]|\, dx.\end{aligned}$$

Following the definitions of $[U_{\theta,\Delta x}^\varepsilon]$ and $[U_{\theta,\Delta x}]$, we obtain

$$\begin{aligned}|[U_{\theta,\Delta x}^\varepsilon] - [U_{\theta,\Delta x}]| &\leqslant |U_{\theta,\Delta x}^\varepsilon(x, j\Delta t^+) - U_{\theta,\Delta x}(x, j\Delta t^+)| \\ &\quad + |U_{\theta,\Delta x}^\varepsilon(x, j\Delta t^-) - U_{\theta,\Delta x}(x, j\Delta t^-)|\end{aligned}$$

and

$$|U_{\theta,\Delta x}^\varepsilon(x, j\Delta t^+) - U_{\theta,\Delta x}(x, j\Delta t^+)| = |U_{\theta,\Delta x}(x \pm \varepsilon\Delta x, j\Delta t^+) - U_{\theta,\Delta x}(x, j\Delta t^+)|$$

(The method of choosing $x\ \pm\varepsilon\Delta x$ depends on the right or left of $m\Delta x$.) The modified approximate solution $U_{\theta,\Delta x}^\varepsilon(x, j\Delta t^+)$ is given by moving the waves in $U_{\theta,\Delta x}(x, j\Delta t^+)$ to the left or right, so these two approximate solutions are different only at some finite number of subintervals of $[(m-1)\Delta x, (m+1)\Delta x]$, and the length for each subinterval is $\varepsilon\Delta x$. Therefore,

$$\begin{aligned}&\int_{(m-1)\Delta x}^{(m+1)\Delta x} |U_{\theta,\Delta x}^\varepsilon(x, j\Delta t^+) - U_{\theta,\Delta x}(x, j\Delta t^+)|\, dx \\ &\quad \leqslant Const \cdot (\varepsilon\Delta x) \cdot \{TV \text{ of } U_{\theta,\Delta x}\ in\ [(m-1)\Delta x, (m+1)\Delta x] \text{ on } E_j\}.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\int_{(m-1)\Delta x}^{(m+1)\Delta x} |U_{\theta,\Delta x}^\varepsilon(x, j\Delta t^-) - U_{\theta,\Delta x}(x, j\Delta t^-)|\, dx \\ &\quad \leqslant Const \cdot (\varepsilon\Delta x) \cdot \{TV\ of\ U_{\theta,\Delta x} \text{ in } [(m-1)\Delta x, (m+1)\Delta x] \text{ on } E_{j-1}\}.\end{aligned}$$

By these two inequalities above, we have

$$\begin{aligned}|J_\varepsilon^j - J^j| &\leqslant C\|\phi\|_\infty(\varepsilon\Delta x)\cdot TV\{U_{\theta,\Delta x}\}\\ &\leqslant Const\cdot\|\phi\|_\infty(\varepsilon(\Delta x)).\end{aligned}$$

This implies that

$$\|J_\varepsilon^j(\theta,\Delta x,\phi) - J^j(\theta,\Delta x,\phi)\|_{L^1}\leqslant Const\cdot\|\phi\|_\infty(\varepsilon(\Delta x)).$$

From previous analysis, it leads to the following theorem for our estimation for $\langle J_\varepsilon^i(\theta,\Delta x,\phi), J_\varepsilon^j(\theta,\Delta x,\phi)\rangle$.

Theorem 4.5. *Given a function ϕ with compact support and piecewise constant between $(m-1)\Delta x$ and $(m+1)\Delta x$ for each time step E_i, $m+i$ is even. If $i\neq j$, then*

$$\langle J_\varepsilon^i(\theta,\Delta x,\phi), J_\varepsilon^j(\theta,\Delta x,\phi)\rangle = O(1)\cdot(\|\phi\|_\infty)^2\cdot(\varepsilon(\Delta x)^2).$$

Here $\langle,\rangle$ is the L^2 inner product on probability space Φ of random number θ.

By Theorems 4.4 and 4.5, we have the following theorem:

Theorem 4.6. *Let $\{U_{\theta,\Delta x}\}$ be the approximate solution of the Cauchy problem* (3) *by the Glimm scheme, and $\{U_{\theta,\Delta x}^\varepsilon; 0<\varepsilon<1\}$ be the modified approximate solution from $\{U_{\theta,\Delta x}\}$. Then for any $0<\varepsilon<1$ we can find a null set $N_\varepsilon\subset\Phi$ and a subsequence $\{\Delta x_i\}\to 0$ such that for any $\theta\in\Phi/N_\varepsilon$ and $\phi\in C_0^1(t>0)$, we have*

$$J_\varepsilon(\theta,\Delta x_i,\phi) = O(1)\cdot\varepsilon^{\frac{1}{2}}\quad \textit{as } \Delta x_i\to 0.$$

Proof. The idea of the proof is from the homogeneous case [2,20]. First given a $0<\varepsilon<1$, let ϕ be a function with compact support and piecewise constant in $[(m-1)\Delta x,(m+1)\Delta x]$ on each time step E_i; $m+i$ is even. Then

$$\begin{aligned}\|J_\varepsilon(\theta,\Delta x_i,\phi)\|_{L^2}^2 &= \left\langle \sum_i J_\varepsilon^i(\theta,\Delta x_i,\phi), \sum_j J_\varepsilon^j(\theta,\Delta x_i,\phi)\right\rangle\\ &= \sum_{k\in\Lambda}\|J_\varepsilon^k(\theta,\Delta x_i,\phi)\|_{L^2}^2 + \sum_{i,j\in\Lambda,i\neq j}\langle J_\varepsilon^i, J_\varepsilon^j\rangle\\ &\leqslant \sum_{k\in\Lambda}\|J_\varepsilon^k\|_{L^\infty}^2 + \sum_{i,j\in\Lambda,i\neq j}\langle J_\varepsilon^i, J_\varepsilon^j\rangle.\end{aligned}$$

By Theorems 4.4 and 4.5, we have

$$\|J_\varepsilon\|_{L^2}^2 \leqslant O(1)\sum_{k\in\Lambda}(\Delta x_i)^2 + O(1)\sum_{i,j\in\Lambda, i\neq j}\varepsilon(\Delta x_i)^2.$$

Since $\sharp\{\Lambda\} = O(1)\frac{r(\phi)}{\Delta t_i}$, we have $\|J_\varepsilon\|_{L^2}^2 \leqslant O(1)(\Delta x_i) + O(1)(\varepsilon)$. This implies that for any ϕ with compact support and piecewise constant, there exists a sequence $\{\Delta x_i\} \to 0$ such that

$$\|J_\varepsilon\|_{L^2} = O(1)\cdot(\varepsilon)^{\frac{1}{2}} \quad \text{as } \Delta x_i \to 0.$$

Next for each $\phi \in L^\infty \cup C_0$, we have

$$\|J_\varepsilon(\cdot, \Delta x_i, \phi)\|_{L^2} \leqslant \|J_\varepsilon(\cdot, \Delta x_i, \phi)\|_{L^\infty} \leqslant O(1)\|\phi\|_{L^\infty}.$$

Let $\{\phi_v\}$ be a sequence of functions with compact support, piecewise constant, and L^∞-dense in the space of test functions. Then for each ϕ_v, there is a sequence $\{\Delta x_i\} \to 0$ such that

$$\|J_\varepsilon(\theta, \Delta x_i, \phi_v)\|_{L^2} = O(1)\cdot(\varepsilon)^{\frac{1}{2}} \quad \text{as } \Delta x_i \to 0.$$

This means that for any ϕ_v, we can find a null set $N_v^\varepsilon \subset \Phi$ and a subsequence $\{\Delta x_{i_k}\} \to 0$ such that

$$J_\varepsilon(\theta, \Delta x_{i_k}, \phi_v) \to O(1)\cdot(\varepsilon)^{\frac{1}{2}} \quad \text{as } \Delta x_{i_k} \to 0 \quad \text{for } \theta \in \Phi/N_v.$$

Let $N_\varepsilon = \cup_v N_\varepsilon^v$; then by the diagonal process we can find a subsequence $\{\Delta\tilde{x}_{i_k}\}$ of $\{\Delta x_{i_k}\}$ such that for each ϕ_v

$$J_\varepsilon(\theta, \Delta\tilde{x}_{i_k}, \phi_v) \to O(1)\cdot(\varepsilon)^{\frac{1}{2}} \quad \text{as } \Delta\tilde{x}_{i_k} \to 0 \quad \text{for } \theta \in \Phi/N_\varepsilon.$$

Now let $\tilde{\phi}$ be a test function, $\tilde{\phi} = (\tilde{\phi} - \phi_v) + \phi_v$, and J_ε be linear with respect to ϕ, so

$$\begin{aligned}
|J_\varepsilon(\theta, \Delta\tilde{x}_{i_k}, \tilde{\phi})| &= |J_\varepsilon(\theta, \Delta\tilde{x}_{i_k}, \tilde{\phi} - \phi_v) + J_\varepsilon(\theta, \Delta\tilde{x}_{i_k}, \phi_v)| \\
&\leqslant |J_\varepsilon(\theta, \Delta\tilde{x}_{i_k}, \tilde{\phi} - \phi_v)| + |J_\varepsilon(\theta, \Delta\tilde{x}_{i_k}, \phi_v)| \\
&\leqslant O(1)\|\tilde{\phi} - \phi_v\|_{L^\infty} + O(1)\cdot(\varepsilon)^{\frac{1}{2}}
\end{aligned}$$

as $\Delta x_{i_k} \to 0$, for $\theta \in \Phi/N_\varepsilon$. Since $\{\phi_v\}$ is L^∞-dense in the space of test functions, we can find v such that $\|\tilde{\phi} - \phi_v\|_{L^\infty}$ is very small. It means that we can find a subsequence

$\{\Delta\tilde{x}_{i_k}\}$ of $\{\Delta x_i\}$ and a null set $N_\varepsilon \subset \Phi$ such that for $\theta \in \Phi/N_\varepsilon$ and $\phi \in C_0^1(t>0)$ we have

$$J_\varepsilon(\theta, \Delta\tilde{x}_{i_k}, \phi) \to O(1)\cdot(\varepsilon)^{\frac{1}{2}} \qquad \text{as } \Delta\tilde{x}_{i_k} \to 0.$$

We complete the proof. □

For the Cauchy problem (3), the main issue in the proof that the limit function $U(x,t)$ is a weak solution of Cauchy problem (3) concerns the convergence of the source term $\{a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x})\}$. Indeed, in our approximate scheme, a_ε is only a pointwise, not a C^1 approximation of a, and thus it follows that a'_ε does not converge to a' in L^1_{loc}. However, we show that if a is Lipschitz-continuous, then $a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x})$ converges weakly (that is, by oscillation) to $a'G(U)$, and thus integrals of $a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x})$ converge to integrals of $a'G(U)$. This is enough to imply that the residual of limit function $U(x,t)$ is zero, and hence $U(x,t)$ is a weak solution of Cauchy problem (3) for almost any choice of sampling.

In the following, we will study the convergence property of $\{a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x})\}$ which leads to the existence of the weak solution of Cauchy problem (3). First we introduce some notation. We let ϕ_δ denote the standard mollifier, and let $G_\delta(U)$ denote a mollification of $G(U)$, that is, define

$$G_\delta(U) \equiv G(U) * \phi_\delta, \tag{28}$$

where "$*$" means the convolution. We also denote $a^\varepsilon_{\theta,\triangle x} \equiv a_\varepsilon$ and the function $U(x,t)$ is as described in Theorem 4.2. The main problem in showing that the residual $R_\phi(U)=0$ lies in showing that

$$\int\int_{t>0} |(a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x}) - a'G(U))\phi|$$

tends to zero as ε, $\Delta x \to 0$. For this, we analyze as follows:

First, we have

$$\begin{aligned}\int\int_{t>0} |(a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x}) - a'G(U))\phi| &\leqslant \left|\int\int_{t>0} (a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x}) - a'G(U^\varepsilon_{\theta,\triangle x}))\phi\right| \\ &\quad + \int\int_{t>0} |a'G(U^\varepsilon_{\theta,\triangle x}) - a'G(U)|\cdot\|\phi\|_{L^\infty} \\ &\equiv I_1 + I_2.\end{aligned}$$

By the Lipschitz continuity of a, there exists a constant K_1 such that

$$I_2 \leqslant K_1 \cdot \int\int_{t>0} |G(U^\varepsilon_{\theta,\triangle x}) - G(U)|. \tag{29}$$

For the term I_1, we have

$$\begin{aligned} I_1 \leqslant & \int\int_{t>0} |a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x}) - a'_\varepsilon G_\delta(U^\varepsilon_{\theta,\triangle x})| \cdot \|\phi\|_{L^\infty} \\ & + \left|\int\int_{t>0} (a'_\varepsilon G_\delta(U^\varepsilon_{\theta,\triangle x}) - a' G_\delta(U^\varepsilon_{\theta,\triangle x}))\phi\right| \\ & + \int\int_{t>0} |a' G_\delta(U^\varepsilon_{\theta,\triangle x}) - a' G(U^\varepsilon_{\theta,\triangle x})| \cdot \|\phi\|_{L^\infty}, \\ \equiv & \; Q_1 + Q_2 + Q_3. \end{aligned}$$

To estimate the term Q_1, since a_ε is a *Lipschitz continuous* mollification of the pointwise constant function $a(x) \equiv a(x_i)$, $x_i \leqslant x \leqslant x_{i+1}$, it follows that there exists a constant $\bar{K}_2$ such that

$$|a'_\varepsilon| \leqslant \frac{\bar{K}_2}{\varepsilon}. \tag{30}$$

This implies that

$$Q_1 \leqslant \frac{K_2}{\varepsilon} \cdot \int\int_{t>0} |G(U^\varepsilon_{\theta,\triangle x}) - G_\delta(U^\varepsilon_{\theta,\triangle x})| \tag{31}$$

for $K_2 = \bar{K}_2 \cdot \|\phi\|_{L^\infty}$. Also, by (28), we have

$$Q_3 \leqslant K_1 \cdot \int\int_{t>0} |G_\delta(U^\varepsilon_{\theta,\triangle x}) - G(U^\varepsilon_{\theta,\triangle x})|. \tag{32}$$

Next, by the construction of G_δ, there exists a constant C such that

$$\int\int_{t>0} |(G(U) - G_\delta(U))\phi| \leqslant C\delta, \tag{33}$$

for any function U in L^1_{loc}. So from (30) to (32), there are two constants K_3 and K_4 such that

$$Q_1 \leqslant \frac{K_3 \ \delta}{\varepsilon}, \tag{34}$$

$$Q_3 \leqslant K_4 \ \delta. \tag{35}$$

Next we consider the term Q_2. By the Lipschitz continuity of a and a_ε again, we have

$$\int\int_{t>0} a'_\varepsilon G_\delta \phi = -\int\int_{t>0} a_\varepsilon (G_\delta \phi)' + \int a_\varepsilon G_\delta(x,0)\phi(x,0)\,dx,$$
$$\int\int_{t>0} a' G_\delta \phi = -\int\int_{t>0} a (G_\delta \phi)' + \int a G(x,0)\phi(x,0)\,dx.$$

It follows that

$$\begin{aligned}
Q_2 &\equiv \left|\int\int_{t>0} (a'_\varepsilon G_\delta(U^\varepsilon_{\theta,\triangle x}) - a' G_\delta(U^\varepsilon_{\theta,\triangle x}))\phi\right| \\
&= \left|\int\int_{t>0} a(G_\delta\phi)'(U^\varepsilon_{\theta,\triangle x}) - \int\int_{t>0} a_\varepsilon(G_\delta\phi)'(U^\varepsilon_{\theta,\triangle x})\right. \\
&\quad \left. + \int a_\varepsilon G_\delta(x,0)\phi(x,0) - aG(x,0)\phi(x,0)\right| \\
&\leqslant \|a-a_\varepsilon\|_\infty \int\int_{t>0} |(G_\delta\phi)'| + \bar{K}_7\|a-a_\varepsilon\|_\infty \int |G_\delta(x,0) - G(x,0)| \\
&\leqslant K_5\cdot(\Delta x)\int\int_{t>0} |G'_\delta| + K_6\cdot\Delta x + K_7\Delta x\cdot\delta \\
&\leqslant \frac{K_8\cdot\Delta x}{\delta} + K_9\cdot\Delta x
\end{aligned} \tag{36}$$

for some constants K_5, K_6, K_7, K_8 and K_9. We will use the above analysis to prove that $R_\phi(U)$=0, thus demonstrating the existence of a solution to the Cauchy problem (3).

To start, consider first the *Residual* $R_\phi(U)$ defined in (26). In the following we wish to prove $R_\phi(U) \to 0$ as ε, $\Delta x \to 0$.

$$|R_\phi(U)| \leqslant |R_\phi(U^\varepsilon_{\theta,\triangle x})| + |R_\phi(U) - R_\phi(U^\varepsilon_{\theta,\triangle x})| \tag{37}$$
$$\leqslant C_1\varepsilon^{\frac12} + C_2\Delta x + \int\int_{t>0} |F(U) - F(U^\varepsilon_{\theta,\triangle x})||\phi_x|$$
$$+\int\int_{t>0} |U - U^\varepsilon_{\theta,\triangle x}||\phi_t| + \int\int_{t>0} |a'_\varepsilon G(U^\varepsilon_{\theta,\triangle x}) - a'G(U)||\phi| \tag{38}$$

for some constants C_1 and C_2. Here the first two terms of (38) estimate $R_\phi(U^\varepsilon_{\theta,\Delta x})$, as shown in the proof of Theorem 4.6. From Theorems 4.2 and 4.3, together with (28), (33) to (36), we can find constants C_3, K_{10}, K_{11}, K_{12} and K_{13} such that

$$|\, R_\phi(U)| \leqslant C_1\varepsilon^{\frac12} + C_3\Delta x + K_{10}\|U^\varepsilon_{\theta,\triangle x} - U\|_{L^1_{\mathrm{loc}}} + \frac{K_{11}\delta}{\varepsilon} + K_{12}\delta + K_{13}\frac{\Delta x}{\delta}. \tag{39}$$

Now to estimate $R_\phi(U)$, fix $\tilde{\varepsilon} > 0$. Choose ε, $\bar{\Delta x}$ small enough such that

$$C_1\varepsilon^{\frac{1}{2}} + C_3\bar{\Delta x} + K_{10}\|U^{\varepsilon}_{\theta,\Delta^- x} - U\|_{L^1_{\mathrm{loc}}} < \frac{\tilde{\varepsilon}}{2}. \tag{40}$$

Then choose $\delta, \Delta x$ such that $\delta < \frac{\varepsilon\tilde{\varepsilon}}{3K_{11}+3K_{12}\varepsilon}$ and $\Delta x < \max\{\bar{\Delta x}, \frac{\tilde{\varepsilon}\delta}{6K_{13}}\}$, so that

$$\frac{K_{11}\delta}{\varepsilon} + K_{12}\delta + K_{13}\frac{\Delta x}{\delta} < \frac{\tilde{\varepsilon}}{2}. \tag{41}$$

Inserting (40) and (41) into (39), we obtain $R_\phi(U) < \tilde{\varepsilon}$. Since $\tilde{\varepsilon}$ is arbitrary, it follows that $R_\phi(U) = 0$.

We have the following main theorem of this paper.

Main Theorem. *Consider the following Cauchy problem of a strictly hyperbolic system*:

$$\begin{cases} a_t = 0, \\ u_t + f(a,u)_x = a' g(a,u), \\ (a,u)(x,0) = (a_0(x), u_0(x)). \end{cases}$$

Assume that $a(x)$ is a Lipschitz-continuous function of x with a bounded total variation, $(a_0(x), u_0(x)) \in L^\infty$, and assume that $TV((a_0(x), u_0(x))$ is sufficiently small. Assume that f and g are smooth vector functions of (a,u) that satisfy

$$\left(\frac{\partial f}{\partial u}\right)^{-1}\left(g - \frac{\partial f}{\partial a}(a,u)\right) \neq 0$$

for all (a,u). Let $\{U^{\varepsilon}_{\theta,\Delta x}\}$ be the modified Glimm scheme approximate solution described in Section 4; *then, there exists a null set $N \in \Phi$ and a sequence $\{\Delta x_i\} \to 0$ such that*

$$\lim_{\Delta x_i \to 0, \varepsilon \to 0} U^{\varepsilon}_{\theta,\Delta x_i} = U(x,t),$$

where $U(x,t)$ is a weak solution of the Cauchy problem.

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